

TORSION IN AN INCOMPLETE TORE

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W. F. CALLAWAY

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TORSION IN AN INCOMPLETE TORE

An approximate solution for the stress distribution in a circular ring sector under uniform torsion using energy methods

by

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Lieutenant Commander, United States Navy

Submitted in partial fulfillment
of the requirements
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THE SUBJECT

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June 1913

TABLE OF CONTENTS

	Page
Introduction.....	1
Formulation of the Problem.....	3
First Approximation.....	8
Second Approximation.....	11
Results.....	16
Conclusions.....	20
References.....	21
Appendix A	22
Appendix B	24

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INTRODUCTION

The stress distribution in an incomplete tore loaded as shown in Fig. 1 is of particular interest since it very closely approximates that in heavy close-coiled helical springs under axial tension or compression. Necessarily the spring helix angle must be small, which is the case in a close-coiled spring. By a heavy spring is meant one whose ratio of mean diameter to cross-sectional diameter is such a value that the curvature of the section must be considered.

It should be noted that the stress distribution arising from the loading in Fig 1. is not pure torsion in the usual sense, but is a combination of torsion and direct shear. The problem therefore resolves itself into one of

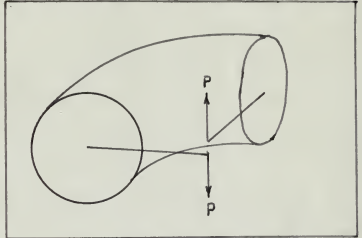


Fig. 1.

finding a single stress function which defines the true stress distribution in a cross-section of the circular ring sector.

Several solutions to the problem are in the literature, all of which by various means solve the differential equation arising from the conditions of compatibility. The first, by Michell (1) in 1899, used polynomial stress functions and obtained solutions for approximately circular cross-sections. Göhner (2) used successive approximations to approach an exact solution. Shepherd (3) used a method similar to both Göhner and Michell by finding a sequence of functions for approximately circular cross-sections and combining them linearly in such a manner that the sum was a solution.

Wahl (4) obtained a solution using curved bar theory and assuming a displacement of the center of rotation. Southwall (5) presented a formal solution for an arbitrary cross-section with a view towards a "relaxation" approach. Frieberger (6) has presented an exact solution for a circular cross-section by finding a stress function analogous to the ordinary torsion function and solving the problem in toroidal harmonics.

In this paper an approximate solution is obtained using the principle of least work. A stress function is found satisfying the equations of equilibrium and the boundary conditions and whose corresponding stresses make the strain energy a minimum. The solution of the differential equation of compatibility has therefore been replaced by the problem of minimizing the strain energy. In the energy method, the condition of minimum strain energy is equivalent to satisfying compatibility not in a point by point sense, but "on the average" throughout the body.

The purpose of this investigation has been to answer two questions in the author's mind. Namely, in view of the fact that nowhere was the author able to find the energy method used in the literature:

- (1) Can the problem be solved by this method, and how do the results compare with those of other solutions?
- (2) Does the problem particularly lend itself to solution by energy methods?

It was found that the problem is not adaptable to an exact solution by energy methods, but by making some approximations, excellent results are obtained that agree very closely with Frieberger's exact solution.

and (4) certain 2 meters only moved but they did assume a dis-

placement at the center of rotation. (5) presented a third set-

ting for an arbitrary origin-motion with a view towards a "rotation"

apex. (6) has presented an exact solution for a circular

rotation-motion by finding a series of points common to the ordinary system

rotation and finding the points in common between.

In this paper an approximate solution is obtained using the principle

of least error. A series of points is found satisfying the equation of motion

and the energy equation and these corresponding points are the

points of energy a minimum. The solution of the differential equation of motion

is then used to obtain a series of points of motion of motion the energy

energy. In the energy method, the solution of motion is used to

obtain a series of points of motion and a point of least error, and

for the energy method throughout the body.

The purpose of this investigation has been to answer the question in

the energy method. Finally, it has to be said that the energy method is the

only method used in the literature;

(1) The energy method is used in the literature, and how to the energy

method also use of other methods?

(2) How the energy method is used in the literature?

energy method?

It was found that the problem is not solvable in an exact solution by

energy method, but by using some approximations, excellent results are

obtained that agree very closely with the energy method's exact solution.

FORMULATION OF THE PROBLEM

We will consider a sector of a circular ring with mean radius of curvature \underline{R} and cross-sectional radius \underline{a} . A load \underline{P} is applied to one terminal cross-section as shown in Fig. 2, the other remaining fixed. Cylindrical coordinates are used, where the \underline{z} axis coincides with the toroidal axis, and the axis of the ring sector lies in the $\underline{r\theta}$ plane.

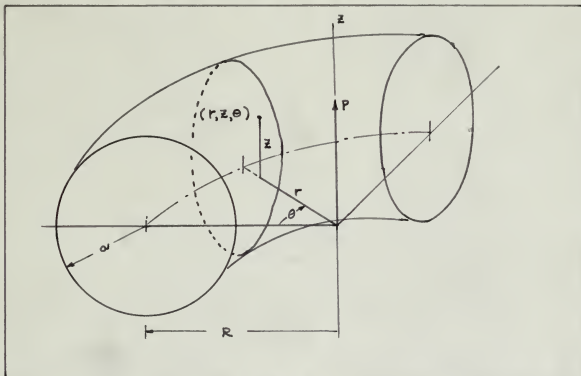


Fig. 2.

θ increases positively as shown in the figure and r increases outward from the toroidal axis. Later in the solution the coordinates will be transformed, but for the present purpose of establishing a stress function satisfying the equations of equilibrium, cylindrical coordinates are most convenient.

Assuming zero body forces, from THEORY OF ELASTICITY, Timoshenko & Goodier, Equations (170) the differential equations of equilibrium are

$$(1) \begin{cases} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ \frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{T_{rz}}{r} = 0 \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2 T_{r\theta}}{r} = 0 \end{cases}$$

Using the same assumptions made by Göhner in this case, namely that the only non-vanishing stresses are $T_{r\theta}$, $T_{z\theta}$ and that the stress distribution in any cross-section is independent of θ these reduce to

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2 T_{r\theta}}{r} = 0$$

This may also be written

$$\left[\frac{\partial}{\partial r} (r^2 T_{r\theta}) + \frac{\partial}{\partial z} (r^2 T_{z\theta}) \right] = 0$$

A stress function ϕ satisfying the above is

$$GR^2 \frac{\partial \phi}{\partial z} = r^2 T_{r\theta} \quad GR^2 \frac{\partial \phi}{\partial r} = -r^2 T_{z\theta}$$

Where G is a constant (actually the modulus of rigidity).

Therefore the stresses may be expressed as

$$(2) \quad T_{r\theta} = \frac{GR^2}{r^2} \frac{\partial \phi}{\partial z} \quad \text{and} \quad T_{z\theta} = -\frac{GR^2}{r^2} \frac{\partial \phi}{\partial r}$$

At this point it is convenient to transform the cylindrical coordinates r, θ, z into toroidal coordinates ρ, ψ, θ (refer to Fig. 3).

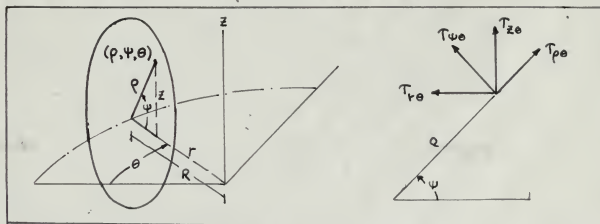


Fig. 3.

If ϕ is a function of r and z , where

$$\begin{aligned} r &= R - \rho \cos \psi \\ r &= \rho \sin \psi \end{aligned}$$

from Fig. 3.

Then

$$\frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \rho} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \rho}$$

$$\frac{\partial \phi}{\partial \psi} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \psi} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \psi}$$

where

$$\frac{\partial r}{\partial \rho} = -\cos \psi$$

$$\frac{\partial r}{\partial \psi} = \rho \sin \psi$$

$$\frac{\partial z}{\partial \rho} = \sin \psi$$

$$\frac{\partial z}{\partial \psi} = \rho \cos \psi$$

Substituting

$$\frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial r} (-\cos \psi) + \frac{\partial \phi}{\partial z} (\sin \psi)$$

$$\frac{\partial \phi}{\partial \psi} = \frac{\partial \phi}{\partial r} (\rho \sin \psi) + \frac{\partial \phi}{\partial z} (\rho \cos \psi)$$

Solving for $\frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial z}$

$$(3) \quad \begin{cases} \frac{\partial \phi}{\partial r} = \left(\frac{\sin \psi}{\rho} \right) \frac{\partial \phi}{\partial \psi} - (\cos \psi) \frac{\partial \phi}{\partial \rho} \\ \frac{\partial \phi}{\partial z} = \left(\frac{\cos \psi}{\rho} \right) \frac{\partial \phi}{\partial \psi} + (\sin \psi) \frac{\partial \phi}{\partial \rho} \end{cases}$$

In a plane cross-section determined by θ a constant

$$(4) \quad \begin{cases} T_{\rho\theta} = -T_{r\theta} \cos \psi + T_{z\theta} \sin \psi \\ T_{\psi\theta} = T_{r\theta} \sin \psi + T_{z\theta} \cos \psi \end{cases}$$

Using Equations (2), (3) and (4) the following result is obtained.

$$\begin{aligned} T_{\rho\theta} &= -\frac{GR^2}{(R-\rho \cos \psi)^2} \left[\frac{\cos \psi}{\rho} \frac{\partial \phi}{\partial \psi} + \sin \psi \frac{\partial \phi}{\partial \rho} \right] - \frac{GR^2 \sin \psi}{(R-\rho \cos \psi)^2} \left[\frac{\sin \psi}{\rho} \frac{\partial \phi}{\partial \psi} - \cos \psi \frac{\partial \phi}{\partial \rho} \right] \\ T_{\psi\theta} &= \frac{GR^2}{(R-\rho \cos \psi)^2} \left[\frac{\cos \psi}{\rho} \frac{\partial \phi}{\partial \psi} + \sin \psi \frac{\partial \phi}{\partial \rho} \right] - \frac{GR^2 \cos \psi}{(R-\rho \cos \psi)^2} \left[\frac{\sin \psi}{\rho} \frac{\partial \phi}{\partial \psi} - \cos \psi \frac{\partial \phi}{\partial \rho} \right] \end{aligned}$$

Reducing

$$(5) \quad T_{\rho\theta} = -\frac{GR^2}{(R-\rho \cos \psi)^2} \frac{1}{\rho} \frac{\partial \phi}{\partial \psi}$$

$$\text{and } T_{\psi\theta} = \frac{GR^2}{(R-\rho \cos \psi)^2} \frac{\partial \phi}{\partial \rho}$$

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$$\psi_{2009} - \gamma = \gamma$$

$$\psi_{1129} = \gamma$$

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2007

$$\psi_{1129} = \frac{\gamma}{\psi_6}$$

$$\psi_{200} = \frac{\gamma}{\psi_6}$$

$$\psi_{2009} = \frac{\gamma}{\psi_6}$$

$$\psi_{112} = \frac{\gamma}{\psi_6}$$

$$\frac{\gamma}{\psi_6} \frac{\psi_6}{\psi_6} + \frac{\gamma}{\psi_6} \frac{\psi_6}{\psi_6} = \frac{\psi_6}{\psi_6}$$

$$\frac{\gamma}{\psi_6} \frac{\psi_6}{\psi_6} + \frac{\gamma}{\psi_6} \frac{\psi_6}{\psi_6} = \frac{\psi_6}{\psi_6}$$

2007

$$(\psi_{112}) \frac{\psi_6}{\psi_6} + (\psi_{200}) \frac{\psi_6}{\psi_6} = \frac{\psi_6}{\psi_6}$$

$$(\psi_{2009}) \frac{\psi_6}{\psi_6} + (\psi_{1129}) \frac{\psi_6}{\psi_6} = \frac{\psi_6}{\psi_6}$$

2007

$$\left. \begin{aligned} \frac{\psi_6}{\psi_6} (\psi_{200}) - \frac{\psi_6}{\psi_6} \left(\frac{\psi_{112}}{\gamma} \right) &= \frac{\psi_6}{\psi_6} \\ \frac{\psi_6}{\psi_6} (\psi_{112}) + \frac{\psi_6}{\psi_6} \left(\frac{\psi_{200}}{\gamma} \right) &= \frac{\psi_6}{\psi_6} \end{aligned} \right\} \quad (2)$$

2007

$$\left. \begin{aligned} \psi_{112} e^{\gamma T} + \psi_{200} e^{\gamma T} &= e^{\gamma T} \\ \psi_{200} e^{\gamma T} + \psi_{112} e^{\gamma T} &= e^{\gamma T} \end{aligned} \right\} \quad (1)$$

2007

$$\left[\frac{\psi_6}{\psi_6} \psi_{200} - \frac{\psi_6}{\psi_6} \frac{\psi_{112}}{\gamma} \right] \frac{\psi_{112} \gamma}{(\psi_{2009} - \gamma)} - \left[\frac{\psi_6}{\psi_6} \psi_{112} + \frac{\psi_6}{\psi_6} \frac{\psi_{200}}{\gamma} \right] \frac{\gamma}{(\psi_{2009} - \gamma)} = e^{\gamma T}$$

$$\left[\frac{\psi_6}{\psi_6} \psi_{200} - \frac{\psi_6}{\psi_6} \frac{\psi_{112}}{\gamma} \right] \frac{\psi_{200} \gamma}{(\psi_{2009} - \gamma)} - \left[\frac{\psi_6}{\psi_6} \psi_{112} + \frac{\psi_6}{\psi_6} \frac{\psi_{200}}{\gamma} \right] \frac{\gamma}{(\psi_{2009} - \gamma)} = e^{\gamma T}$$

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$$\frac{\psi_6}{\psi_6} \frac{\gamma}{(\psi_{2009} - \gamma)} = e^{\gamma T}$$

2007

$$\frac{\psi_6}{\psi_6} \frac{1}{\gamma} \frac{\gamma}{(\psi_{2009} - \gamma)} = e^{\gamma T} \quad (2)$$

The latter expressions relate the stress function and the stresses in the new system of coordinates.

It follows that since the shear stress $\tau_{\rho\theta}$ is normal to the boundary, it must vanish everywhere on the boundary. This is true because the surface of the body is free from any external forces. Using this condition with Equation (5), it is apparent that $\frac{\partial \phi}{\partial \psi} = 0$ and ϕ must be constant on the boundary.

The circular ring sector we are considering is a singly connected body, hence the constant may be chosen arbitrarily. Therefore the boundary condition is taken as $\phi = 0$ everywhere on the boundary.

The only action on a cross-section is a force \underline{P} directed along the toroidal axis. This may be resolved into a force and a couple as shown in Fig. 4.

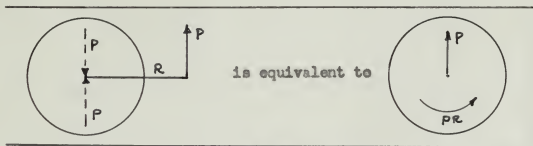


Fig. 4.

It is now seen that the two conditions of static equilibrium to be satisfied are that the resultant stress on a cross-section produce a force \underline{P} directed along the \underline{z} axis and a moment about the center \underline{PR} . These requirements may be written

$$(6) \left\{ \begin{array}{l} P = \int_0^a \int_0^{2\pi} (\tau_{\rho\theta} \sin \psi + T_{\psi\theta} \cos \psi) \rho d\rho d\psi \\ PR = \int_0^a \int_0^{2\pi} T_{\psi\theta} \rho^2 d\rho d\psi \end{array} \right.$$

The latter expression relates the stress function and the stresses in

the two sides of the boundary.

It follows that since the shear stress $\tau_{\theta\phi}$ is normal to the boundary,

it must satisfy everywhere on the boundary. There is also assumed the relation

of the body is free from any external forces. Being this condition with

Equation (2), it is apparent that $\frac{\partial \phi}{\partial \psi} = 0$ and ϕ must be constant on the boundary.

Therefore,

The circular ring section we are considering is a simply connected body,

hence the boundary can be chosen arbitrarily. Therefore the boundary condition

is taken as $\phi = 0$ everywhere on the boundary.

The ring section in a cross-section is a force P directed along the

vertical axis. This may be resolved into a force and a couple as shown in

Fig. 4.

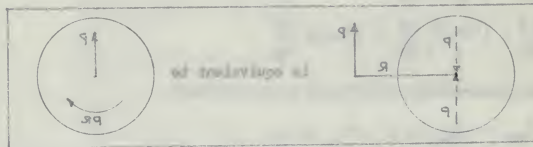


Fig. 4.

It is now seen that the two conditions of static equilibrium can be

realized and that the boundary stress on a cross-section involves a force P

directed along the ψ axis and a moment about the origin O . These requirements

may be written

$$\left\{ \begin{aligned} PR &= \int_0^{2\pi} \int_0^1 T_{\theta\phi} r^2 d\phi d\psi \\ P &= \int_0^{2\pi} \int_0^1 (T_{\theta\phi} r \sin \psi + T_{\phi\theta} r \cos \psi) r d\phi d\psi \end{aligned} \right. \quad (5)$$

The strain energy per unit angle θ is

$$(7) \text{---} \quad U = \frac{1}{2G} \int_0^a \int_0^{2\pi} (\tau_{\rho\theta}^2 + \tau_{\psi\theta}^2) (R - \rho \cos \psi) \rho \, d\rho \, d\psi$$

The method of solution will now be to take the stress function in the form

$$\phi = \sum_{i=0}^N \alpha_i \phi_i, \quad \text{where } \phi_i \text{ are suitably selected functions of } \rho \text{ and}$$

ψ , each of which satisfies the boundary condition $\phi_i = 0$ when $\rho = a$.

The coefficients α_i are constants which are evaluated from the minimum condition of strain energy.

FIRST APPROXIMATION

For a first approximation we shall take a function ϕ , satisfying the boundary condition that it vanish everywhere on the boundary, in the form $\phi = (\rho^2 - a^2)(\alpha_0 + \frac{\alpha_1 \rho}{R} \cos \psi)$. The reasons for this particular choice are discussed in Appendix A. Taking the partial derivatives of ϕ with respect to the two variables ρ and ψ

$$\frac{\partial \phi}{\partial \rho} = 2\rho\alpha_0 + \frac{\alpha_1(3\rho^2 - a^2)}{R} \cos \psi \quad \text{and} \quad \frac{\partial \phi}{\partial \psi} = -\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi$$

Substituting in Equations (5), the following expressions are obtained for $T_{\rho\theta}$ and $T_{\psi\theta}$.

$$(8) \text{---} T_{\rho\theta} = \frac{GR^2}{(R - \rho \cos \psi)^2} \left[\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi \right] \text{ and } T_{\psi\theta} = \frac{GR^2}{(R - \rho \cos \psi)^2} \left[2\rho\alpha_0 + \frac{\alpha_1(3\rho^2 - a^2)}{R} \cos \psi \right]$$

The appearance of the term $\frac{1}{(R - \rho \cos \psi)^2}$ in the stress equations makes the integration required in (6) and (7) very complicated and the results largely unmanageable in the evaluation of the unknown coefficients in ϕ . (See Appendix B). This is particularly true when additional terms are used in ϕ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{\rho}{R}$ is always less than unity, we may write

$$\frac{R^2}{(R - \rho \cos \psi)^2} = \frac{1}{(1 - \frac{\rho}{R} \cos \psi)^2} = 1 + 2\left(\frac{\rho}{R}\right) \cos \psi + 3\left(\frac{\rho}{R}\right)^2 \cos^2 \psi + \dots$$

Utilizing this expansion, the exact stress expressions (5) may be approximated as follows

$$T_{\rho\theta} = -\frac{G}{\rho} \left[\left(1 + 2\frac{\rho}{R} \cos \psi\right) \frac{\partial \phi_0}{\partial \psi} \alpha_0 + \frac{\partial \phi_1}{\partial \psi} \alpha_1 \right]$$

$$T_{\psi\theta} = G \left[\left(1 + 2\frac{\rho}{R} \cos \psi\right) \frac{\partial \phi_0}{\partial \rho} \alpha_0 + \frac{\partial \phi_1}{\partial \rho} \alpha_1 \right]$$

This particular form of approximation accomplishes the desired result

For a first approximation we shall take a function ϕ , satisfying the boundary condition that it is zero everywhere on the boundary, in the form

$$\phi = (q^2 - a^2)(\omega_0 + \frac{a}{R} q^2 \cos \psi)$$

The reason for this

particular choice has already been explained in Appendix I. Taking the partial derivatives of ϕ with respect to the two variables ψ and q

$$\frac{\partial \phi}{\partial \psi} = 2aq\omega_0 + \frac{a^2}{R}(q^2 - a^2) \cos \psi \quad \text{and} \quad \frac{\partial \phi}{\partial q} = -\frac{2a}{R}(q^2 - a^2) \cos \psi$$

substituting in formulas (1), the following expressions are

obtained for T_ϕ and T_ψ .

$$(8) \quad T_\phi = \frac{2R^2}{R - q \cos \psi} \left[\frac{2aq\omega_0}{R} + \frac{a^2}{R}(q^2 - a^2) \cos \psi \right] \quad \text{and} \quad T_\psi = \frac{2R^2}{R - q \cos \psi} \left[\frac{2a}{R}(q^2 - a^2) \cos \psi \right]$$

The appearance of the term $\frac{1}{(R - q \cos \psi)^2}$ in the above expressions

makes the integration required in (6) and (7) very complicated and the results largely meaningless in the evaluation of the unknown coefficients in ϕ . (See Appendix B). This is particularly true when additional terms are used in ϕ for a higher order of approximation, and in the evaluation of the strain energy where the stresses appear as squared terms.

Since $\frac{q}{R}$ is always less than unity, we may write

$$\frac{1}{(1 - \frac{q}{R} \cos \psi)^2} = 1 + 2\left(\frac{q}{R}\right) \cos \psi + 3\left(\frac{q}{R}\right)^2 \cos^2 \psi + \dots$$

Utilizing this expansion, the exact stress expressions (2) may be

approximated as follows

$$T_\phi \approx -\frac{2}{9} \left[\left(1 + 2\frac{q}{R} \cos \psi \right) \frac{\partial \phi}{\partial \psi} + \omega_0 + \frac{a}{R} \frac{\partial \phi}{\partial q} \right] \quad \text{and} \quad T_\psi \approx 2 \left[\left(1 + 2\frac{q}{R} \cos \psi \right) \frac{\partial \phi}{\partial \psi} + \omega_0 + \frac{a}{R} \frac{\partial \phi}{\partial q} \right]$$

This particular form of approximation considerably simplifies the resulting results

of limiting the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in the stress equations.

Since $\phi_0 = (\rho^2 - a^2)$ and $\phi_1 = \frac{\rho(\rho^2 - a^2)}{R} \cos \psi$

The partial derivatives are

$$\begin{aligned} \frac{\partial \phi_0}{\partial \rho} &= 2\rho & \frac{\partial \phi_1}{\partial \rho} &= \frac{(3\rho^2 - a^2)}{R} \cos \psi \\ \frac{\partial \phi_0}{\partial \psi} &= 0 & \frac{\partial \phi_1}{\partial \psi} &= -\frac{\rho(\rho^2 - a^2)}{R} \sin \psi \end{aligned}$$

Substituting, we arrive at the following approximate expressions for the stresses.

$$(9) \text{-----} \begin{cases} \tau_{\rho\theta} = G \left[\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi \right] \\ \tau_{\psi\theta} = G \left[2\rho\alpha_0 + \frac{(4\rho^2\alpha_0 + 3\rho^2\alpha_1 - a^2\alpha_1)}{R} \cos \psi \right] \end{cases}$$

Substituting these values of $\tau_{\rho\theta}$ and $\tau_{\psi\theta}$ in the first of Equations (6), and integrating we obtain

$$P = \frac{G\pi a^4}{R} \alpha_0 \quad \therefore \alpha_0 = \frac{PR}{G\pi a^4}$$

The same result is obtained from the second condition of Equations (6).

It follows that α_0 is fixed by the requirements of static equilibrium and α_1 may now be determined by the condition of minimum strain energy that $\frac{\partial U}{\partial \alpha_1} = 0$.

From Equation (7)

$$\frac{\partial U}{\partial \alpha_1} = \frac{1}{G} \int_0^a \int_0^{2\pi} \left(\tau_{\rho\theta} \frac{\partial \tau_{\rho\theta}}{\partial \alpha_1} + \tau_{\psi\theta} \frac{\partial \tau_{\psi\theta}}{\partial \alpha_1} \right) (R - \rho \cos \psi) \rho d\rho d\psi$$

Substituting the stresses from Equation (9) and integrating

$$\frac{\partial U}{\partial \alpha_1} = \frac{R\pi}{G} \left[\left(\frac{1}{2} \frac{a^4}{R^2} \right) \alpha_0 + \left(\frac{2}{3} \frac{a^4}{R^2} \right) \alpha_1 \right]$$

Setting $\frac{\partial U}{\partial \alpha_1} = 0$ and solving for α_1 ,

of limited the system power to within the period $\frac{1}{R}$ and $\frac{1}{R}$ is chosen to be
 some constant.

$$\psi_0 = \phi \quad \text{and} \quad \psi_1 = \phi \frac{(q - \frac{1}{2})}{R} \cos \psi$$

The initial conditions are

$$\begin{aligned} \psi_{20} &= \frac{\phi}{96} & \psi_{21} &= \frac{\phi}{96} \\ \psi_{20} &= \frac{\phi}{96} & \psi_{21} &= \frac{\phi}{96} \end{aligned}$$

Integrating, we arrive at the following approximate expressions for

$$\left. \begin{aligned} \psi_{20} &= \frac{\phi}{96} \left[\cos \psi - \frac{(q - \frac{1}{2})}{R} \sin \psi \right] \\ \psi_{21} &= \frac{\phi}{96} \left[\cos \psi + \frac{(q - \frac{1}{2})}{R} \sin \psi \right] \end{aligned} \right\} \quad (9)$$

Substituting these values of ψ_{20} and ψ_{21} in the three equations (5),

and integrating we obtain

$$\psi = \frac{2\pi a}{R} \omega \quad \therefore \quad \omega = \frac{R}{2\pi a} \psi$$

The same result is obtained from the second condition in Equation (5).

It follows that ω is linear in the components of vector multiplication

and ω , but may be determined by the condition of constant strain energy

$$\frac{\partial U}{\partial \omega} = 0$$

from Equation (7)

$$\psi \left(\frac{\partial U}{\partial \omega} \right) = \frac{1}{2} \int_0^{2\pi} \left(\frac{\partial U}{\partial \omega} \right) d\psi = 0$$

Carrying out the integration from Equation (7) and integrating

$$\left[\frac{1}{2} \left(\frac{\partial U}{\partial \omega} \right) + \left(\frac{\partial U}{\partial \omega} \right) \right] \frac{\pi R}{2} = \frac{\partial U}{\partial \omega}$$

$$\omega = \frac{\partial U}{\partial \omega}$$

Using these results in Equations (9) we arrive at the expressions for the first approximation of the stress distribution in a cross-section of the incomplete tore

$$(10) \quad \left\{ \begin{array}{l} \tau_{\rho\theta} = - \frac{PR}{\pi a^4} \left[\frac{3}{4} \frac{(\rho^2 - a^2)}{R} \sin \psi \right] \\ \tau_{\psi\theta} = \frac{PR}{\pi a^4} \left[2\rho + \frac{(\rho^2 + 3a^2)}{R} \cos \psi \right] \end{array} \right.$$

At the point of maximum stress where $\rho = a$ and $\psi = 0$ the above reduce to

$$(11) \quad \left\{ \begin{array}{l} \tau_{\rho\theta} = 0 \\ \left[\tau_{\psi\theta} \right]_{\max} = \frac{2PR}{\pi a^3} \left[1 + \frac{5}{4} \left(\frac{a}{R} \right) \right] \end{array} \right.$$

It is interesting to note at this point that for this particular solution, one of the unknown coefficients in ϕ is determined directly from the requirements of static equilibrium, and the other directly from the minimum strain energy condition without constraint arising from static equilibrium.

From these results in addition (9) is true in the approximation
for the first approximation of the space distribution in a wave packet
of the linearized form

$$\left\{ \begin{aligned} T_{\theta} &= -\frac{pR}{\pi\omega} \left[\frac{\varepsilon}{4} \frac{(q^2 - \omega^2)}{R} \sin\psi \right] \\ T_{\psi} &= \frac{pR}{\pi\omega} \left[\frac{1}{2} q + \frac{(Tq^2 + 3\omega^2)}{R} \cos\psi \right] \end{aligned} \right. \quad (10)$$

At the point of interest where $\psi = \alpha$ and $\psi = 0$ the above

$$\left\{ \begin{aligned} T_{\theta} &= 0 \\ T_{\psi} &= \left[T_{\psi} \right]_{\psi=\alpha} = \frac{2pR}{\pi\omega} \left[1 + \frac{\varepsilon}{4} \left(\frac{\alpha}{R} \right) \right] \end{aligned} \right. \quad (11)$$

It is interesting to note in this case that for this particular solution,
one of the unknown coefficients in ϕ is determined directly from the periodic
nature of wave equations, and the other directly from the relation between
energy conditions which connects with wave equations.

SECOND APPROXIMATION

A closer approximation to the true stress conditions will result if higher order terms of a suitable nature are used in the stress function. We shall now take ϕ as

$$\phi = (\rho^2 - a^2) \left(\alpha_0 + \frac{\alpha_1 \rho}{R} \cos \psi + \frac{\alpha_2 \rho^2}{R^2} \cos^2 \psi + \frac{\alpha_3 \rho^2}{R^2} \sin^2 \psi + \frac{\alpha_4 a^2}{R^2} \right)$$

Reasons for this particular choice of functions are discussed in Appendix A.

Again employing the binomial expansion of $\frac{1}{(R - \rho \cos \psi)}$ we write approximate expressions for $T_{\rho\theta}$ and $T_{\psi\theta}$.

$$T_{\rho\theta} = -\frac{G}{\rho} \left[\left(1 + 2\frac{\rho}{R} \cos \psi + 3\frac{\rho^2}{R^2} \cos^2 \psi \right) \frac{\partial \phi_0}{\partial \psi} \alpha_0 + \left(1 + 2\frac{\rho}{R} \cos \psi \right) \frac{\partial \phi}{\partial \psi} \alpha_1 + \frac{\partial \phi_2}{\partial \psi} \alpha_2 + \frac{\partial \phi_3}{\partial \psi} \alpha_3 + \frac{\partial \phi_4}{\partial \psi} \alpha_4 \right]$$

$$T_{\psi\theta} = G \left[\left(1 + 2\frac{\rho}{R} \cos \psi + 3\frac{\rho^2}{R^2} \cos^2 \psi \right) \frac{\partial \phi_0}{\partial \rho} \alpha_0 + \left(1 + 2\frac{\rho}{R} \cos \psi \right) \frac{\partial \phi}{\partial \rho} \alpha_1 + \frac{\partial \phi_2}{\partial \rho} \alpha_2 + \frac{\partial \phi_3}{\partial \rho} \alpha_3 + \frac{\partial \phi_4}{\partial \rho} \alpha_4 \right]$$

Where

$$\begin{aligned} \phi_0 &= (\rho^2 - a^2) & \phi_2 &= \frac{\rho^2(\rho^2 - a^2)}{R^2} \cos^2 \psi & \phi_4 &= \frac{a^2(\rho^2 - a^2)}{R^2} \\ \phi_1 &= \frac{\rho(\rho^2 - a^2)}{R} \cos \psi & \phi_3 &= \frac{\rho^2(\rho^2 - a^2)}{R^2} \sin^2 \psi \end{aligned}$$

This is an extension of the device used before to limit the highest power to which the ratios $\frac{a}{R}$ and $\frac{\rho}{R}$ appear in each term of the stress equations. Since $\frac{a}{R}$ and $\frac{\rho}{R}$ occur in a like manner in ϕ_2 , ϕ_3 and ϕ_4 , these latter terms are grouped together and treated in similar fashion when introduced into the approximate expressions for the stresses.

Taking the partial derivatives, substituting and rearranging the terms for convenient integration, the following approximate expressions for $T_{\rho\theta}$ and $T_{\psi\theta}$ are obtained.

$$(12) \left\{ \begin{aligned} T_{\rho\theta} &= G \left[\frac{\alpha_1(\rho^2 - a^2)}{R} \sin \psi + \frac{2\rho(\alpha_1 + \alpha_2 - \alpha_3)(\rho^2 - a^2)}{R^2} \sin \psi \cos \psi \right] \\ T_{\psi\theta} &= G \left[\left[\frac{6\rho^2 \alpha_0}{R^2} + \frac{2\rho \alpha_1(3\rho^2 - a^2)}{R^2} + \frac{2\rho \alpha_2(2\rho^2 - a^2)}{R^2} \right] \cos^2 \psi + \left[\frac{4\rho^2 \alpha_0}{R} + \frac{\alpha_1(3\rho^2 - a^2)}{R} \right] \cos \psi + 2\rho \alpha_0 + \right. \\ &\quad \left. \frac{2\rho \alpha_3(2\rho^2 - a^2)}{R^2} \sin^2 \psi + \frac{2a^2 \alpha_4 \rho}{R^2} \right] \end{aligned} \right\}$$

It is assumed that the ...
 It is assumed that the ...
 It is assumed that the ...

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial t^2} \right) = \phi$$

Because the ...
 This ...

$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial t^2} \right] = \phi$$

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial t^2} \right) = \phi$$

This is an ...
 power to which the ...
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 these latter ...
 introduced into the ...

Using the ...
 for convenient ...

$$\left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial t^2} \right\} = \phi$$

From the first of the static equilibrium conditions in (6) (that the resultant stress must produce a force \underline{P} in the \underline{Z} direction) it is again found that $\alpha_0 = \frac{PR}{G\pi a^4}$.

The second static equilibrium condition (that the resultant stress must produce a moment about the center equal to \underline{PR}) gives the following result.

$$\frac{PR}{G\pi a^4} = \left[\left(1 + \frac{a^2}{R^2}\right)\alpha_0 + \left(\frac{1}{2} \frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4 \right]$$

However, since $\alpha_0 = \frac{PR}{G\pi a^4}$

$$\left(\frac{1}{2} \frac{a^2}{R^2}\right)\alpha_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\alpha_3 + \left(\frac{a^2}{R^2}\right)\alpha_4 + \left(\frac{a^2}{R^2}\right)\left(\frac{PR}{G\pi a^4}\right) = 0$$

Since $\alpha_1, \alpha_2, \alpha_3$ and α_4 will ultimately all contain the factor $\frac{PR}{G\pi a^4}$ some simplification of the algebra will be afforded if we make the following substitutions

$$\beta_n = \alpha_n \left(\frac{G\pi a^4}{PR} \right) \quad \text{where } n = 1, 2, 3, 4$$

Finally the constraining function derived from the conditions of static equilibrium to be used in minimizing the strain energy is

$$(13) \text{-----} \left(\frac{1}{2} \frac{a^2}{R^2}\right)\beta_1 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\beta_2 + \left(\frac{1}{6} \frac{a^2}{R^2}\right)\beta_3 + \left(\frac{a^2}{R^2}\right)\beta_4 + \frac{a^2}{R^2} = 0$$

The work involved in obtaining the partial derivatives of the strain energy with respect to the unknown coefficients $\alpha_1, \alpha_2, \alpha_3$ and α_4 will be simplified by differentiating under the integral sign.

Therefore

$$\frac{\partial U}{\partial \alpha_n} = \frac{R}{G} \int_0^a \int_0^{2\pi} \left(T_{\rho\theta} \frac{\partial T_{\rho\theta}}{\partial \alpha_n} + T_{\psi\theta} \frac{\partial T_{\psi\theta}}{\partial \alpha_n} \right) \left(1 - \frac{\rho}{R} \cos \psi \right) \rho d\rho d\psi$$

$$\frac{\partial U}{\partial \alpha_0} \quad \text{is not required since } \alpha_0 \text{ has already been evaluated from the}$$

From the time of the 1960s, the Commission has been

where \mathbf{u}_i is the i th coordinate of \mathbf{u} and $\mathbf{u}_i^2 = u_i u_i$ = element-wise squares.

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The second step is evaluation (and the resulting plans

and further a number of the same type of the following

• *Idioms*

[illegible]

$$\frac{99}{200} = 0.495$$

[illegible]

These 2,1,3,6-tetra-O-acetyl-2,6-diamino-2,3,5-trideoxy- α -D-glucopyranosides will ultimately all contain the factor

some indication of the degree of the effect it is not possible to

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$$4\epsilon\epsilon\epsilon\epsilon = N \quad \left(\frac{4\pi\epsilon}{89}\right) N^2 = 2$$

Finally, the following formulae are derived from the definition of \mathcal{A} :

apart from the fact that it is not a very good idea to have a large number of small, separate, uncoordinated efforts.

$$\left(\frac{1}{5}R_5\right)^2 + \left(\frac{1}{5}R_5\right)^2 + \left(\frac{1}{5}R_5\right)^2 + \left(\frac{1}{5}R_5\right)^2 + \left(\frac{1}{5}R_5\right)^2 = 0$$

The next involved an operation the person believed of the victim

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be simplified by differentiating under the integral sign.

04/01/2007

$$\psi_k q b q \left(\psi \cos \frac{\pi}{2} - 1 \right) \left(\frac{\partial \psi T}{\partial \psi} \psi T + \frac{\partial \psi T}{\partial \psi} \psi T \right) \left(\frac{\partial \psi T}{\partial \psi} \psi T + \frac{\partial \psi T}{\partial \psi} \psi T \right) \left(\frac{\partial \psi T}{\partial \psi} \psi T + \frac{\partial \psi T}{\partial \psi} \psi T \right)$$

It was pointed out that the above information was already being reviewed by the

conditions of static equilibrium. Since $\frac{\partial U}{\partial \beta_n} = (\text{Constant}) \frac{\partial U}{\partial \beta_n}$, it is convenient here to take $\frac{\partial U}{\partial \beta_n}$. After substituting for $T_{\rho\theta}$ and $T_{\psi\theta}$ from Equations (12) and performing the required integration, the following expressions are obtained.

$$(14) \left\{ \begin{aligned} \frac{\partial U}{\partial \beta_1} &= \frac{\pi a^4}{GR} \left[\left(\frac{3}{2} - \frac{5}{16} \frac{a^2}{R^2} \right) + \frac{2}{3} \beta_1 + \frac{13}{48} \frac{a^2}{R^2} \beta_1 + \frac{1}{16} \frac{a^2}{R^2} \beta_3 + \frac{1}{2} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_2} &= \frac{\pi a^4}{GR} \left[\left(\frac{1}{3} - \frac{1}{4} \frac{a^2}{R^2} \right) + \frac{13}{16} \frac{a^2}{R^2} \beta_1 + \frac{7}{24} \frac{a^2}{R^2} \beta_1 + \frac{1}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_3} &= \frac{\pi a^4}{GR} \left[\left(\frac{1}{3} + \frac{1}{12} \frac{a^2}{R^2} \right) + \frac{1}{16} \frac{a^2}{R^2} \beta_1 + \frac{1}{24} \frac{a^2}{R^2} \beta_2 + \frac{7}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 \right] \\ \frac{\partial U}{\partial \beta_4} &= \frac{\pi a^4}{GR} \left[\left(2 + \frac{2}{3} \frac{a^2}{R^2} \right) + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{3} \frac{a^2}{R^2} \beta_2 + \frac{1}{3} \frac{a^2}{R^2} \beta_3 + 2 \frac{a^2}{R^2} \beta_4 \right] \end{aligned} \right.$$

To minimize the strain energy and evaluate the unknown coefficients $\beta_1, \beta_2, \beta_3$ and β_4 , the method of Lagrangian multipliers will be used with the constraining function (13) established by the requirements of static equilibrium. The constant $\frac{\pi a^4}{GR}$ appearing in the partial derivatives of the strain energy will be incorporated in the multiplier. Letting λ be a Lagrangian multiplier, and $f(\beta_1, \beta_2, \beta_3, \beta_4) = 0$ the constraining function, we may write

$$\frac{\partial U}{\partial \beta_n} + \lambda \frac{\partial f}{\partial \beta_n} = 0$$

$$f(\beta_1, \beta_2, \beta_3, \beta_4) = 0$$

where $n = 1, 2, 3, 4$

Using (13) and (14) in the above, and the fact that

$$\frac{\partial f}{\partial \beta_1} = \frac{1}{2} \frac{a^2}{R^2}, \quad \frac{\partial f}{\partial \beta_2} = \frac{1}{6} \frac{a^2}{R^2}, \quad \frac{\partial f}{\partial \beta_3} = \frac{1}{6} \frac{a^2}{R^2}, \quad \frac{\partial f}{\partial \beta_4} = \frac{a^2}{R^2}$$

$$\frac{UG}{AG} (\text{unbalanced}) = \frac{UG}{AG}$$

the following conditions are satisfied:

$$\left. \begin{aligned} \left[\left(\frac{1}{2} - \frac{1}{16} \frac{2}{3} + \frac{1}{2} \right) \frac{2}{3} \right] &= \frac{U_6}{46} \\ \left[\left(\frac{1}{2} - \frac{1}{16} \frac{2}{3} + \frac{1}{2} \right) \frac{2}{3} \right] &= \frac{U_6}{46} \\ \left[\left(\frac{1}{2} - \frac{1}{16} \frac{2}{3} + \frac{1}{2} \right) \frac{2}{3} \right] &= \frac{U_6}{46} \\ \left[\left(\frac{1}{2} - \frac{1}{16} \frac{2}{3} + \frac{1}{2} \right) \frac{2}{3} \right] &= \frac{U_6}{46} \end{aligned} \right\} \quad (1)$$

To minimize the impact, we will be working with the relevant authorities to ensure that the necessary measures are taken to protect the environment and the safety of the public.

and the other of the same kind as the first, but with the following modifications:

used with the conventional Equation (1) established up the requirements of

relative equilibria. The constant $\frac{\pi}{2R}$ appearing in the period brackets

of the strain energy will be transferred to the initiator.

$$0 = \begin{pmatrix} +\partial & \partial_2 & \partial_1 \\ \partial_2 & \partial_1 & \partial \end{pmatrix} \quad \text{and} \quad \text{a Lagrangian multiplier, and}$$

$$0 = \frac{76}{96} \lambda + \frac{26}{96}$$

$$f \in \mathcal{E}(S, \mathbb{R}) \Rightarrow 0$$

$$0 = (29, 29, 29, 29)^T$$

— *Journal of the American Medical Association*, 1964, 191: 1000.

$$\frac{9}{9} = 1, \quad \frac{9}{9} = 1, \quad \frac{9}{9} = 1, \quad \frac{9}{9} = 1, \quad \frac{9}{9} = 1$$

we arrive at the following set of equations, the solution of which will evaluate $\beta_1, \beta_2, \beta_3, \beta_4$ and determine the stress function.

$$(15) \left\{ \begin{aligned} \left(\frac{3}{2} - \frac{5}{16} \frac{a^2}{R^2} \right) + \frac{2}{3} \beta_1 + \frac{13}{48} \frac{a^2}{R^2} \beta_2 + \frac{1}{16} \frac{a^2}{R^2} \beta_3 + \frac{1}{2} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{2 R^2} \\ \left(\frac{1}{3} - \frac{1}{4} \frac{a^2}{R^2} \right) + \frac{13}{16} \frac{a^2}{R^2} \beta_1 + \frac{7}{24} \frac{a^2}{R^2} \beta_2 + \frac{1}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{6 R^2} \\ \left(\frac{1}{3} + \frac{1}{12} \frac{a^2}{R^2} \right) + \frac{1}{16} \frac{a^2}{R^2} \beta_1 + \frac{1}{24} \frac{a^2}{R^2} \beta_2 + \frac{7}{24} \frac{a^2}{R^2} \beta_3 + \frac{1}{3} \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{6 R^2} \\ \left(2 + \frac{2}{3} \frac{a^2}{R^2} \right) + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{3} \frac{a^2}{R^2} \beta_2 + \frac{1}{3} \frac{a^2}{R^2} \beta_3 + 2 \frac{a^2}{R^2} \beta_4 &= -\frac{\lambda' a^2}{R^2} \\ \frac{a^2}{R^2} + \frac{1}{2} \frac{a^2}{R^2} \beta_1 + \frac{1}{6} \frac{a^2}{R^2} \beta_2 + \frac{1}{6} \frac{a^2}{R^2} \beta_3 + \frac{a^2}{R^2} \beta_4 &= 0 \end{aligned} \right.$$

where $\lambda' = \left(\frac{GR}{\pi a^3} \right) \lambda$

Solving for $\beta_1, \beta_2, \beta_3$ and β_4

$$\beta_1 = -\frac{3}{4} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \quad \beta_2 = \frac{57}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] - \frac{56}{96}$$

$$\beta_3 = \frac{8}{96} - \frac{3}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \quad \beta_4 = -\frac{88}{96} + \frac{27}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right]$$

The corresponding values of α are

$$\alpha_1 = -\frac{3}{4} \frac{PR}{G\pi a^4} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \quad \alpha_2 = \frac{PR}{G\pi a^4} \left\{ \frac{57}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] - \frac{56}{96} \right\}$$

$$\alpha_3 = \frac{PR}{G\pi a^4} \left\{ \frac{8}{96} - \frac{3}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \right\} \quad \alpha_4 = \frac{PR}{G\pi a^4} \left\{ -\frac{88}{96} + \frac{27}{96} \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \right\}$$

Using these results in Equations (12), we may now write expressions representing a second approximation of the stress distribution in a cross-section of the incomplete tore.

$$(16) \left\{ \begin{aligned} \tau_{\rho\theta} &= -\frac{PR}{\pi a^4} \left[\frac{3}{4} \delta (\rho^2 - a^2) \frac{\sin \psi}{R} + \left(\frac{2}{3} + \frac{1}{8} \delta \right) \frac{\sin 2\psi}{R^2} \right] \text{ where } \delta = \left[\frac{1 - \frac{37}{72} \frac{a^2}{R^2}}{1 - \frac{43}{192} \frac{a^2}{R^2}} \right] \\ \tau_{\psi\theta} &= \frac{PR}{\pi a^4} \left\{ \left[\left(\frac{10}{3} - 2\delta \right) \rho^3 + \left(\frac{4}{3} + \frac{1}{4} \delta \right) \rho a^2 \right] \frac{\cos^2 \psi}{R^2} + \left[\left(4 - \frac{9}{4} \delta \right) \rho^2 + \left(\frac{3}{4} \delta \right) a^2 \right] \frac{\cos \psi}{R} + \left[\left(\frac{1}{3} - \frac{1}{8} \delta \right) \rho^3 - \left(2 - \frac{5}{8} \delta \right) \rho a^2 \right] \frac{1}{R^2} + 2\rho \right\} \end{aligned} \right.$$

At the point of maximum stress, where $\rho = a$ and $\psi = 0$, the above reduce to

$$(17) \left\{ \begin{aligned} \tau_{\rho\theta} &= 0 \\ [\tau_{\psi\theta}]_{\max} &= \frac{2PR}{\pi a^3} \left[1 + \left(2 - \frac{3}{4} \delta \right) \frac{a}{R} + \left(\frac{3}{2} - \frac{5}{8} \delta \right) \frac{a^2}{R^2} \right] \end{aligned} \right.$$

As is apparent from the foregoing development, further approximations utilizing additional terms in the stress function will result in extremely long and tedious calculations. This in itself is a limitation of this method. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional radius to the mean radius of curvature of the tore in order to compare results with other solutions.

Being these results in Equation (11), we may now write the corresponding a second representation of the above relationship in a more compact form as follows:

$$\left\{ \begin{aligned} & \left[\frac{1 - \frac{1}{2} \frac{\psi}{R}}{1 - \frac{1}{2} \frac{\psi}{R}} \right] \left[\frac{1}{R} \left(\frac{1}{2} - \frac{1}{8} \frac{\psi}{R} + \frac{1}{8} \frac{\psi^2}{R^2} \right) \right] \\ & \left[\frac{1}{R} \left(\frac{1}{2} - \frac{1}{8} \frac{\psi}{R} + \frac{1}{8} \frac{\psi^2}{R^2} \right) \right] \left[\frac{1}{R} \left(\frac{1}{2} - \frac{1}{8} \frac{\psi}{R} + \frac{1}{8} \frac{\psi^2}{R^2} \right) \right] \end{aligned} \right\} \quad (16)$$

is the point of solution above, where $\psi = 0$, the above

$$\left\{ \begin{aligned} & \frac{1}{R} \left(\frac{1}{2} - \frac{1}{8} \frac{\psi}{R} + \frac{1}{8} \frac{\psi^2}{R^2} \right) \\ & \frac{1}{R} \left(\frac{1}{2} - \frac{1}{8} \frac{\psi}{R} + \frac{1}{8} \frac{\psi^2}{R^2} \right) \end{aligned} \right\} \quad (17)$$

as is apparent from the foregoing development, further approximations utilizing additional terms in the stress function will result in extremely long and tedious calculations. This in itself is a indication of the extent. Therefore at this point, assuming the solution to be a rapidly converging one, we will stop and introduce actual values of the ratio of cross-sectional radius to the mean radius of curvature of the face in order to compare results with other solutions.

RESULTS

The distribution of shearing stress on a horizontal diameter is shown in Fig. 5 and the circumferential stress distribution in Fig. 6. In both cases a ratio of R/a equal to 4 is used since this realizes the worst condition (i.e. greatest curvature for a given cross-section) of any practical significance.

The quantity \underline{S} appearing as the ordinate in both curves is a dimensionless quantity and is equal to $\frac{\tau_{\theta\theta} \pi a^2}{P}$, since $\tau_{\theta\theta}$ vanishes on both a horizontal diameter and the periphery. A stress distribution representing pure torsion of a straight circular shaft is shown by a dotted line in both figures. It is seen that the maximum stress actually existing in the core is considerably greater than that derived from ordinary torsion theory. Both curves are in good agreement with similar ones derived from the exact solution by Frieberger. Points from Frieberger's curves appear as the small circles in Fig. 5 and 6.

In comparing the results of this solution with others, namely G hner, Wahl, and Frieberger, the point of maximum stress will be used as a reference with different values of R/a . Table 1 gives values of \underline{K} in the expression

$$[\tau_{\theta\theta}]_{\max} = \frac{2PR}{\pi a^3} [\underline{K}] \quad \text{for the several solutions.}$$

Table 1.

R/a	Exact	This Solution		Other Approx. Solutions	
	Frieberger	1st Approx.	2nd Approx.	G�hner	Wahl
4	1.376	1.313	1.371	1.372	1.400
5	1.293	1.250	1.287	1.295	1.310
6	1.237	1.209	1.234	1.239	1.252
8	1.171	1.156	1.171	1.172	1.184
10	1.136	1.125	1.134	1.135	1.145

The probability of finding a given state in a particular direction is shown in Fig. 2 and the corresponding cross-section in Fig. 3. It will be seen that the value of $\frac{dP}{d\Omega}$ is a function of the angle θ and the direction of the wave vector k . The probability of finding a given state in a particular direction is shown in Fig. 2 and the corresponding cross-section in Fig. 3.

The quantity $\frac{dP}{d\Omega}$ is a function of the angle θ and the direction of the wave vector k . It will be seen that the value of $\frac{dP}{d\Omega}$ is a function of the angle θ and the direction of the wave vector k . The probability of finding a given state in a particular direction is shown in Fig. 2 and the corresponding cross-section in Fig. 3. It will be seen that the value of $\frac{dP}{d\Omega}$ is a function of the angle θ and the direction of the wave vector k . The probability of finding a given state in a particular direction is shown in Fig. 2 and the corresponding cross-section in Fig. 3.

In computing the values of the cross-sections, the value of the wave vector k is used as a reference value. The value of k is given in Table 1. The value of k is given in Table 1.

$$\left[\frac{dP}{d\Omega} \right]_{\theta=0} = \left[\frac{dP}{d\Omega} \right]_{\theta=0} \left[\frac{dP}{d\Omega} \right]_{\theta=0}$$

Table 1

k	dP/dΩ	dP/dΩ		dP/dΩ	
		at θ=0	at θ=π	at θ=0	at θ=π
1	1.170	1.171	1.170	1.170	1.170
2	1.170	1.170	1.170	1.170	1.170
3	1.170	1.170	1.170	1.170	1.170
4	1.170	1.170	1.170	1.170	1.170
5	1.170	1.170	1.170	1.170	1.170
6	1.170	1.170	1.170	1.170	1.170

Table 1. indicates that the energy method applied to this problem produces results which compare favorably with other solutions. It also appears that the solution converges rapidly, since only five terms were used in the stress function.

The following table shows the results of the experiments conducted on the effect of the concentration of the solution on the rate of reaction. The results show that the rate of reaction increases with the concentration of the solution.

The following table shows the results of the experiments conducted on the effect of the concentration of the solution on the rate of reaction. The results show that the rate of reaction increases with the concentration of the solution.

Concentration of Solution (M)	Time taken for reaction to complete (s)	Rate of reaction (1/s)	Concentration of Solution (M)	Time taken for reaction to complete (s)	Rate of reaction (1/s)
0.1	100	0.01	0.2	50	0.02
0.2	50	0.02	0.3	33.3	0.03
0.3	33.3	0.03	0.4	25	0.04
0.4	25	0.04	0.5	20	0.05
0.5	20	0.05	0.6	16.7	0.06

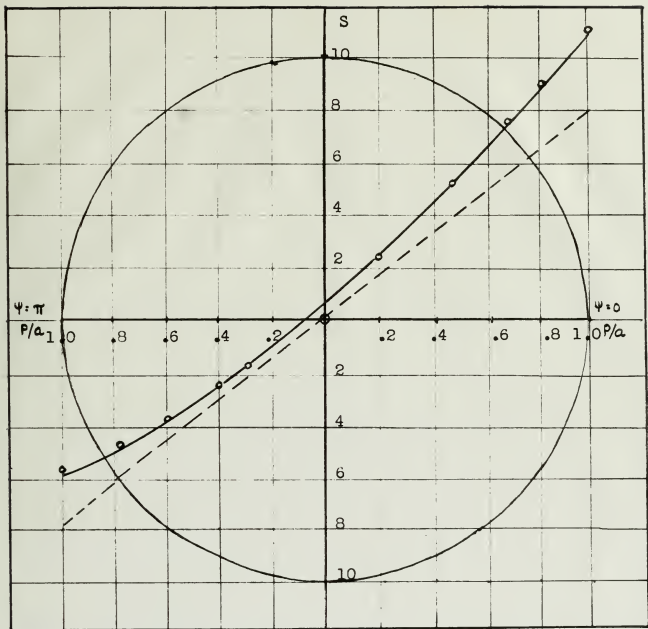


Fig. 5.

Distribution of shearing stress on a horizontal diameter for $a/R=1/4$. ($\psi=0, \pi$). $S=(\tau_{\psi\theta})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

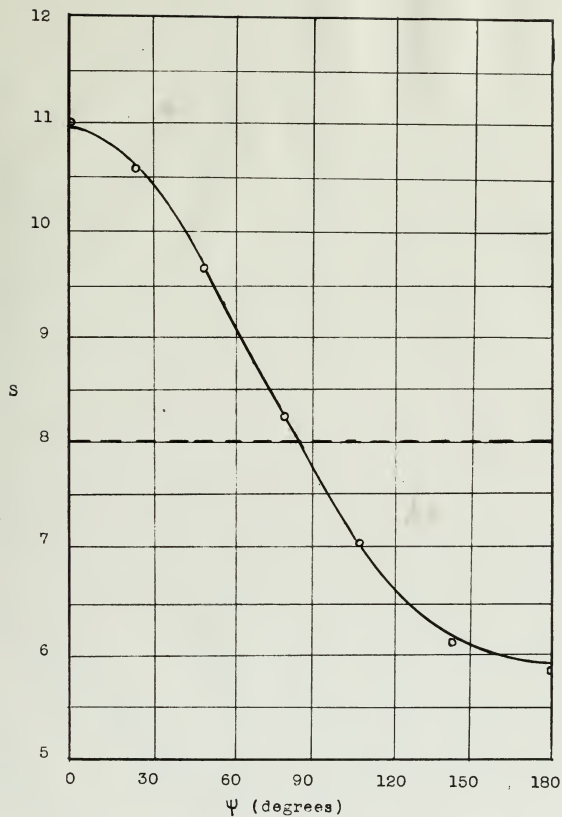


Fig. 6.

Circumferential stress distribution for $a/R=1/4$. $S=(\tau_{\psi\theta})(\pi a^2)/P$. Frieberger's points are indicated by small circles.

CONCLUSIONS

In discussing any conclusions from this investigation, it would be appropriate to recall the two questions that prompted it.

- (1) Can the problem be solved by this method, and how do the results compare with those of other solutions?
- (2) Does the problem particularly lend itself to solution by energy methods?

First, the method will work and acceptable results are obtained with a relatively few terms in the stress function. This is in itself worthy of note, since it allows a very complex problem to be attacked by the more elementary methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real basis for judging the results. The fact also that the energy method requires minimizing an integral, which is done only with extreme difficulty with any number of terms in the stress function, is a limitation to its adaptability.

In conclusion then, it may be said that this solution has the value of arriving at very good results using a relatively uncomplicated stress function of only five terms.

CONCLUSIONS

In discussing any mechanism from this investigation, it would be

appropriate to recall the two questions that prompted it.

(1) Can the problem be solved by this method, and how do the results

compare with those of other solutions?

(2) Does the problem satisfactorily lend itself to solution by energy

methods?

First, the method will work and acceptable results are obtained with a relatively few terms in the stress function. This is in itself worthy of note, since it allows a very complex problem to be attacked by the more elementary

methods of mathematics.

However, in reference to the second question, there are limitations both inherent in the energy method and peculiar to this particular application, that strongly indicate the problem is not especially adapted to a solution by energy methods.

The energy method, except in unusual circumstances, does not provide an exact solution. Consequently, in the absence of an exact solution, there is no real means for judging the results. The fact also that the energy method requires minimizing an integral, which is done only with extreme difficulty when any number of terms in the stress function, is a limitation to its adaptability. In conclusion then, it may be said that this solution has the value of serving as a good example using a relatively uncomplicated stress function of only five terms.

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APPENDIX A

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in general this procedure is too difficult, a limited number N of suitable functions was selected to determine an approximate stress function.

In choosing functions of ρ and ψ in $\phi = \sum_{i=0}^N \alpha_i \phi_i$, the first consideration was the boundary condition $\phi_i = 0$ when $\rho = a$. This condition was satisfied by taking each ϕ_i to contain the factor $(\rho^2 - a^2)$.

Then
$$\phi = (\rho^2 - a^2) \sum_{i=0}^N \alpha_i f_i(\rho, \psi)$$

With rectangular coordinates (ξ, η) in mind, where $\xi = \rho \cos \psi$ and $\eta = \rho \sin \psi$, the next logical step was to express $\sum f_i(\xi, \eta)$ in a power series. The first six terms of such a series were considered, namely those involving

$$1, \xi, \eta, \xi^2, \eta^2, \xi\eta$$

Since a horizontal diameter ($\eta = 0$) on a plane cross-section (θ is a constant) is an axis of symmetry for the ϕ surface, ϕ must be even in η and not contain terms involving odd powers of η . In general ξ will appear to all powers since $(\xi = 0)$ is not axis of symmetry. Therefore the remaining terms expressed as functions of ρ and ψ are

$$1, \rho \cos \psi, \rho^2 \cos^2 \psi, \rho^2 \sin^2 \psi$$

The term $\frac{a^2}{R^2}$ in ϕ_4 while not consistent with this line of reasoning, appeared as a result of the binomial expansion used in approximating the stresses. It was extracted from G6hners solution where a like approximation was used.

According to the principle of least work which is used in this solution, an exact stress function would require selecting from all functions that satisfy the boundary condition those which minimize the strain energy.

Since in General 10's procedure is too difficult, a limited number of suitable functions was selected to determine an approximate stress function.

In choosing functions of ρ and ψ in $\phi = \sum_{n=0}^{\infty} \rho^n \psi_n$, the first consideration was the boundary condition $\phi = 0$ when $\rho = 0$.

This condition was satisfied by taking each ψ_n to contain the factor $(\rho^2 - 1)$. Then
$$\phi = \sum_{n=0}^{\infty} (\rho^2 - 1) \psi_n = 0$$

With rectangular coordinates (r, θ) in mind, where $\rho = r \cos \theta$ and $\psi = r \sin \theta$, the next logical step was to express ϕ in a power series. The first six terms of such a series were considered, namely those involving

$$1, \theta, \theta^2, \theta^3, \theta^4, \theta^5$$

Since a horizontal diameter ($\theta = 0$) on a plane cross-section ϕ is a constant is an axis of symmetry for the ϕ surface, ϕ must be even in θ and not contain terms involving odd powers of θ . In General 10's case θ is not an axis of symmetry. Therefore the remaining terms involve even powers of θ and ψ are

$$1, \psi^2 \cos^2 \theta, \psi^2 \cos^4 \theta, \psi^2 \cos^6 \theta, \dots$$

The term $\frac{\psi^2}{r^2}$ in ϕ will not contribute to the line of resistance, appeared as a result of the binomial expansion used in approximating the stress. It was excluded from the stress solution where a line approximation was used.

Consequently the stress function was taken in the form

$$\phi = (\rho^2 - a^2) \left[\alpha_0 + \alpha_1 \left(\frac{\rho}{R} \right) \cos \psi + \alpha_2 \left(\frac{\rho}{R} \right)^2 \cos^2 \psi + \alpha_3 \left(\frac{\rho}{R} \right)^2 \sin^2 \psi + \alpha_4 \frac{a^2}{R^2} \right]$$

Here ρ was replaced by $\frac{\rho}{R}$ so that α_i would in all cases be the product of a dimensionless number and the factor $\frac{PR}{6\pi a^4}$.

In the first approximation the first two terms were used and in the second all five were introduced in ϕ .

APPENDIX B

The appearance of the term $\frac{1}{(R-\rho \cos \psi)^2}$ in the exact equations relating the stresses and the stress function gives rise to the occurrence of integrals of the type $\int_0^a \int_0^{2\pi} \frac{\rho^m \sin^n \psi \cos^g \psi \, d\rho \, d\psi}{(R-\rho \cos \psi)^2}$ and $\int_0^a \int_0^{2\pi} \frac{\rho^m \sin^n \psi \cos^g \psi \, d\rho \, d\psi}{(R-\rho \cos \psi)^3}$ in evaluating the strain energy and in consideration of the conditions of static equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $g=0$ we have

$$\int_0^a \int_0^{2\pi} \frac{\rho \, d\rho \, d\psi}{(R-\rho \cos \psi)^2}$$

Integrating first with respect to ψ and setting $R=c$ and $-\rho=b$

$$\int \frac{d\psi}{(c+b \cos \psi)^2}$$

Letting

$$P = \frac{\sin \psi}{(c+b \cos \psi)^2}$$

Then

$$\begin{aligned} \frac{dP}{d\psi} &= \frac{\cos \psi (c+b \cos \psi) + b(1-\cos^2 \psi)}{(c+b \cos \psi)^2} = \frac{b+c \cos \psi}{(c+b \cos \psi)^2} \\ &= \frac{b - \frac{c^2}{b} + \frac{c}{b}(c+b \cos \psi)}{(c+b \cos \psi)^2} = \frac{c}{b} \left(\frac{1}{c+b \cos \psi} \right) - \frac{c^2-b^2}{b} \left[\frac{1}{(c+b \cos \psi)^2} \right] \end{aligned}$$

Multiplying by $d\psi$ and integrating

$$\begin{aligned} \int \frac{dP}{d\psi} d\psi = P &= \frac{\sin \psi}{c+b \cos \psi} = \frac{c}{b} \int \frac{d\psi}{c+b \cos \psi} - \frac{c^2-b^2}{b} \int \frac{d\psi}{(c+b \cos \psi)^2} \\ \int \frac{d\psi}{c+b \cos \psi} &= -\frac{b}{c^2-b^2} \left(\frac{\sin \psi}{c+b \cos \psi} \right) + \frac{c}{c^2-b^2} \int \frac{d\psi}{c+b \cos \psi} \\ &= \frac{1}{\sqrt{c^2-b^2}} \cos^{-1} \left(\frac{b+c \cos \psi}{c+b \cos \psi} \right) \quad \text{where } c^2 > b^2 \end{aligned}$$

APPENDIX B

The appearance of the term $\frac{1}{(R - \rho \cos \psi)^2}$ in the strain energy relation
the stresses and the stress function gives rise to the occurrence of integrals
of the type $\int_0^{2\pi} \frac{\rho^{2n} \psi^{2m} \cos^2 \psi \sin^2 \psi}{(R - \rho \cos \psi)^2} d\psi$ and $\int_0^{2\pi} \frac{\rho^{2n} \psi^{2m} \cos^2 \psi \sin^2 \psi}{(R - \rho \cos \psi)^3} d\psi$
evaluating the strain energy and in consideration of the condition of static
equilibrium.

Taking the simplest form of the first type, where $m=1$, $n=0$ and $\delta=0$

$$\text{we have } \int_0^{2\pi} \frac{\psi^2 \sin^2 \psi}{(R - \rho \cos \psi)^2} d\psi$$

Integrating first with respect to ψ and setting $R = c$ and $\rho = d$

$$\int \frac{\psi^2}{(c + d \cos \psi)^2} d\psi$$

Letting

$$\frac{\psi \sin \psi}{(c + d \cos \psi)^2} = \psi$$

Then

$$\frac{d\psi}{\psi} = \frac{c \psi \cos \psi (c + d \cos \psi) + d (1 - \cos^2 \psi)}{(c + d \cos \psi)^2} = \frac{d + c \cos \psi}{(c + d \cos \psi)^2}$$

$$= \frac{d - \frac{c}{d} + \frac{c}{d} (c + d \cos \psi)}{(c + d \cos \psi)^2} = \frac{1}{d} \left(\frac{c}{c + d \cos \psi} - \frac{c^2 - d^2}{d^2} \right)$$

Multiplying by $d\psi$ and integrating

$$\int \frac{d\psi}{\psi} = \psi = \frac{\psi \sin \psi}{c + d \cos \psi} = \frac{c}{d} \left[\frac{d\psi}{c + d \cos \psi} - \frac{c^2 - d^2}{d^2} \right]$$

$$\int \frac{d\psi}{c + d \cos \psi} = \psi \cos \psi = - \frac{d}{c^2 - d^2} \left(\frac{\psi \sin \psi}{c + d \cos \psi} \right) + \frac{c}{c^2 - d^2} \left[\frac{d\psi}{c + d \cos \psi} - \frac{c^2 - d^2}{d^2} \right]$$

$$= \frac{1}{\sqrt{c^2 - d^2}} \cos^{-1} \left(\frac{d + c \cos \psi}{c + d \cos \psi} \right) \text{ where } c > d$$

Therefore

$$\int \frac{d\psi}{(c+b\cos\psi)^2} = -\frac{b}{c^2-b^2} \left(\frac{\sin\psi}{c+b\cos\psi} \right) + \frac{c}{(c^2-b^2)^{3/2}} \cos^{-1} \left[\frac{b+c\cos\psi}{c+b\cos\psi} \right]$$

Introducing the limits 0 and 2π , this reduces to

$$\frac{2\pi c}{(c^2-b^2)^{3/2}} \quad \text{where} \quad \begin{array}{l} c=R \\ b=-\rho \end{array}$$

Therefore

$$\begin{aligned} \int_0^a \int_0^{2\pi} \frac{\rho d\rho d\psi}{(R-\rho\cos\psi)^2} &= 2\pi R \int_0^a \frac{\rho d\rho}{(R^2-\rho^2)^{3/2}} = 2\pi R \left[\frac{1}{(R^2-\rho^2)^{1/2}} \right]_0^a \\ &= 2\pi \left[\frac{R}{(R^2-a^2)^{1/2}} - 1 \right] \end{aligned}$$

The other more complicated forms where $n \neq 0$ and $q \neq 0$ are integrable in finite terms by similar reduction methods, but it is apparent that the work becomes excessively involved. Also the results in the form just developed are not readily usable in evaluating the unknown coefficients in the stress function.

In view of the foregoing, despite the fact that it was not actually necessary, it was expedient to approximate the stresses in such a manner that the integration was simplified and the results put in a usable form.

This device of approximating the stress equations compromised the requirement that the stresses satisfy the equations of equilibrium. However, it appears that, since the stresses do satisfy the conditions of minimum strain energy and static equilibrium, and give satisfactory results, the compromise may be tolerated.

$$\left[\frac{d\psi}{(c+d\cos\psi)^2} \right]_0^{2\pi} = \frac{d}{c^2} \left[\frac{2\pi\psi}{(c+d\cos\psi)} \right]_0^{2\pi} + \frac{c}{(c^2-d^2)^{3/2}} \left[\frac{1+d\cos\psi}{c+d\cos\psi} \right]_0^{2\pi}$$

Introducing the limits 0 and 2π , this becomes

$$\frac{2\pi d}{(c^2-d^2)^{3/2}} = \frac{d}{c^2} \left[\frac{2\pi\psi}{(c+d\cos\psi)} \right]_0^{2\pi} + \frac{c}{(c^2-d^2)^{3/2}} \left[\frac{1+d\cos\psi}{c+d\cos\psi} \right]_0^{2\pi}$$

$$\int_0^{2\pi} \frac{d\psi}{(c+d\cos\psi)^2} = \frac{2\pi R}{(c^2-d^2)^{3/2}} = \frac{2\pi R}{(c^2-d^2)^{3/2}} \left[\frac{1}{(R-d)^2} \right]_0^{2\pi}$$

$$= \frac{2\pi R}{(c^2-d^2)^{3/2}} \left[\frac{1}{(R-d)^2} - \frac{1}{(R+d)^2} \right]$$

The other two integrations have where $\psi = 0$ and $\psi = \pi$ are integrable

in finite terms by similar reduction methods, but it is apparent that the work

becomes excessively involved. Also the results in the form just obtained are

not readily usable in evaluating the unknown coefficients in the stress function.

In view of the foregoing, besides the fact that it was not actually necessary

it was expedient to approximate the stresses in each case by using the integration

was simplified and the results put in a concise form.

This series of approximations for stress approximations was presented in the

last two stresses exactly the conditions of equilibrium. However, it appears that

these the stresses do satisfy the conditions of equilibrium exactly, and finally

equilibrium, and give satisfactory results, the conditions are satisfied.

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